

# ON THE CHRISTOFFEL FUNCTION FOR THE GENERALIZED JACOBI MEASURES ON A QUASIDISK

Vladimir V. Andrievskii

## Abstract

We establish the exact (up to the constants) double inequality for the Christoffel function for a measure supported on a Jordan domain bounded by a quasiconformal curve. We show that this quasiconformality of the boundary cannot be omitted.

## 1. Introduction and Main Results

Denote by  $\mathbf{P}_n$  the set of all complex polynomials of degree at most  $n \in \mathbf{N}_0 := \{0, 1, 2, \dots\}$ . For a finite Borel measure  $\nu$  on the complex plane  $\mathbf{C}$  such that its support is compact and it consists of infinitely many points, and a parameter  $1 \leq p < \infty$ , the  $n$ -th *Christoffel function* associated with  $\nu$  and  $p$ , is defined by

$$(1.1) \quad \lambda_n(\nu, p, z) := \inf_{\substack{p_n \in \mathbf{P}_n \\ p_n(z)=1}} \int |p_n|^p d\nu, \quad z \in \mathbf{C}.$$

This function plays an important role in the theory of orthogonal polynomials, in particular, due to the following *Christoffel Variational Principle* (see [21, p. 78] or [18, p. 309]):

$$(1.2) \quad \lambda_n(\nu, 2, z) = \left( \sum_{j=0}^n |\pi_j(\nu, z)|^2 \right)^{-1}, \quad z \in \mathbf{C},$$

where  $\pi_j(\nu, \cdot)$  is the  $j$ -th orthonormal polynomial associated with measure  $\nu$ .

The starting point of our consideration consists of two groups of results. The first group includes recent findings in [22]-[24] about the behavior of  $\lambda_n(\nu, p, z)$  in the case where  $\nu$  is supported on a Jordan arc or curve. The second group includes results in [20, 1, 2, 12, 19] about the behavior of  $\pi_n(\nu, z)$  in the case of a (weighted) area type measure  $\nu$ . We refer the reader to these papers for the further references.

We consider measures supported on the closure  $\overline{G}$  of a domain  $G \subset \mathbf{C}$  bounded by a Jordan curve  $L := \partial G$ . Let  $\Omega := \overline{\mathbf{C}} \setminus \overline{G}$ , where  $\overline{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$  is the extended

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complex plane. The Riemann mapping function  $\Phi : \Omega \rightarrow \mathbf{D}^* := \{w : |w| > 1\}$  normalized by

$$\Phi(\infty) = \infty, \quad \Phi'(\infty) := \lim_{z \rightarrow \infty} \frac{\Phi(z)}{z} > 0$$

plays an essential role in our consideration, which from this point of view, can be compared with the results in the above mentioned papers.

We focus our attention to the case where  $G$  is a bounded *quasidisk*, i.e.,  $L$  is a bounded *quasiconformal curve* (see [4, 13]) which geometrically means that for every pair of different points  $z_1, z_2 \in L$ ,

$$(1.3) \quad \min(\text{diam } L', \text{diam } L'') \leq C_L |z_1 - z_2|,$$

where  $L'$  and  $L''$  denote the two connected components (subarcs) of  $L \setminus \{z_1, z_2\}$ ,  $\text{diam } S$  is the diameter of a set  $S \subset \mathbf{C}$ , and  $C_L \geq 1$  is a constant depending only on  $L$ .

For fixed  $z_j \in L$  and  $\alpha_j > -2, j = 1, \dots, m$ , consider the *weight function*

$$(1.4) \quad h(z) := \begin{cases} h_0(z) \prod_{j=1}^m |z - z_j|^{\alpha_j} & \text{if } z \in G, \\ 0 & \text{if } z \in \mathbf{C} \setminus G, \end{cases}$$

where, for a measurable function  $h_0$ , the inequality

$$C_h^{-1} \leq h_0(z) \leq C_h, \quad z \in G$$

holds with a constant  $C_h \geq 1$  depending only on  $h$ .

A measure  $\nu$  supported on  $\overline{G}$  and determined by  $d\nu = h dm$ , where  $dm$  stands for the 2-dimensional Lebesgue measure (area) in the plane, is called the *generalized Jacobi measure*.

Let

$$d(z, S) := \text{dist}(\{z\}, S) := \inf_{\zeta \in S} |z - \zeta|, \quad z \in \mathbf{C}, S \subset \mathbf{C},$$

and let for  $\delta > 0$  and  $z \in L$ ,

$$L_\delta := \{\zeta \in \Omega : |\Phi(\zeta)| = 1 + \delta\}, \quad \rho_\delta(z) := d(z, L_\delta).$$

**Theorem 1** *Let  $G$  be a quasidisk,  $\nu$  be the generalized Jacobi measure, and let  $1 \leq p < \infty$ . Then for  $n \in \mathbf{N} := \{1, 2, \dots\}$  and  $z \in L$ ,*

$$(1.5) \quad C^{-1} \leq \lambda_n(\nu, p, z) \rho_{1/n}(z)^{-2} \prod_{j=1}^m (|z - z_j| + \rho_{1/n}(z))^{-\alpha_j} \leq C$$

*holds with  $C = C(G, h, p) \geq 1$ .*

According to (1.2) and (1.5), for the orthogonal polynomials  $\pi_n(\nu, z)$  and  $z \in L$ , we have

$$(1.6) \quad \begin{aligned} |\pi_n(\nu, z)| &\leq \lambda_n(\nu, 2, z)^{-1/2} \\ &\leq C^{1/2} \rho_{1/n}(z)^{-1} \prod_{j=1}^m (|z - z_j| + \rho_{1/n}(z))^{-\alpha_j/2}. \end{aligned}$$

Using (1.6) and well-known distortion properties of conformal mappings with quasiconformal extension (see [15]) one can obtain more specialized bounds for orthogonal polynomials which can be found, for example, in [20, 1, 2] where they are proved by other methods.

If  $G = \mathbf{D} := \{z : |z| < 1\}$  and  $d\nu = h_0 dm$ , then (1.6) becomes

$$(1.7) \quad |\pi_n(\nu, z)| \leq C^{1/2} n, \quad z \in \overline{\mathbf{D}}.$$

Keeping in mind the Rakhmanov's [16] solution of the Steklov problem (for more details, see [8] or [10]), it is tempting to conjecture that (1.7) as well as (1.6) cannot be improved.

The inequality (1.5) can also be used to estimate  $|\pi_n(\nu, z)|$  from below. For example, if  $\alpha_j = 0$ , i.e.,  $h(z) = h_0(z)$  for all  $z \in G$ , then, by virtue of (1.5), for any quasidisk  $G$  and  $1 \leq p < \infty$  we have

$$(1.8) \quad C^{-1} \rho_{1/n}(z)^2 \leq \lambda_n(\nu, p, z) \leq C \rho_{1/n}(z)^2, \quad z \in L, n \in \mathbf{N},$$

which, together with (2.13) below, imply that there exists  $k = k(G) \in \mathbf{N} \setminus \{1\}$  such that for  $z \in L$  and  $n \in \mathbf{N}$ ,

$$\sum_{j=n+1}^{kn} |\pi_j(\nu, z)|^2 = \lambda_{kn}(\nu, 2, z)^{-1} - \lambda_n(\nu, 2, z)^{-1} \geq \frac{1}{2C \rho_{1/(kn)}(z)^2}.$$

Therefore,

$$\max_{n < j \leq kn} |\pi_j(\nu, z)| \geq \frac{\varepsilon}{\sqrt{n} \rho_{1/(kn)}(z)}, \quad \varepsilon := (2kC)^{-1/2},$$

that is,

$$\max_{n < j \leq kn} (\sqrt{j} \rho_{1/j}(z) |\pi_j(\nu, z)|) \geq \varepsilon,$$

which yields that for any  $z \in L$  there exists an infinite set  $\Lambda_z \subset \mathbf{N}$  such that

$$(1.9) \quad |\pi_n(\nu, z)| \geq \frac{\varepsilon}{\sqrt{n} \rho_{1/n}(z)}, \quad n \in \Lambda_z.$$

Note that the case  $h(z) \equiv 1$  on  $G = \mathbf{D}$  shows the exactness of (1.9) (up to the constant  $\varepsilon$ ).

Next, consider the domain

$$G^* := \{z = x + iy : 0 < x < 1, |y| < e^{-1/x}\}$$

which is obviously not a quasidisk.

**Theorem 2** *For the area measure  $m^*$  supported on  $\overline{G^*}$ ,  $1 \leq p < \infty$ , and  $k \in \mathbf{N}$ ,*

$$(1.10) \quad \lim_{n \rightarrow \infty} \frac{\lambda_n(m^*, p, 0)}{\rho_{1/n}(0)^k} = 0.$$

Comparing the left-hand side of (1.8) and (1.10) shows that the requirement in Theorem 1 on  $G$  to be a quasidisk cannot be omitted.

Using the approach from the proof of [5, Theorem 2] or [24, Corollary 2.5] the same inequality (1.5) can be proved if  $G$  is replaced by a finite union of quasidisks lying exterior to each other. We do not dwell on this purely technical problem.

The structure of this paper is as follows. Section 2 contains auxiliary results from theory of quasiconformal mappings and constructive function theory in the complex plane. In Section 3, we prove the main results, i.e., Theorem 1 and Theorem 2.

In what follows, we always assume that  $G$  is a quasidisk and  $h$  is a generalized Jacobi measure. We use the convention that  $c, c_1, \dots$  denote positive constants and  $\varepsilon, \varepsilon_1, \dots$  sufficiently small positive constants (different in different sections). If not stated otherwise, we assume that these constants can depend only on  $G, p$ , and  $h$ . For the nonnegative functions  $a$  and  $b$  we write  $a \preceq b$  if  $a \leq cb$ , and  $a \asymp b$  if  $a \preceq b$  and  $b \preceq a$  simultaneously.

We complete this section with the additional notation:

$$D(z, r) := \{\zeta : |\zeta - z| < r\}, \quad C(z, r) := \{\zeta : |\zeta - z| = r\}, \quad z \in \mathbf{C}, r > 0.$$

## 2. Auxiliary Results and Constructions

We begin with estimation of two integrals.

**Lemma 1** *Let  $\delta > 0, \alpha > -2, \beta > 2 + |\alpha|$ . Then for  $z', z'' \in \mathbf{C}$  we have*

$$(2.1) \quad I := \int_{\mathbf{C}} (|\zeta - z''| + \delta)^{-\beta} |\zeta - z'|^\alpha dm(\zeta) \leq c_1 \delta^{2-\beta} (|z' - z''| + \delta)^\alpha,$$

where  $c_1 = c_1(\alpha, \beta)$ .

**Proof.** Consider two particular cases.

If  $|z' - z''| \leq \delta$ , then using the polar coordinates with center at  $z'$ , we obtain

$$\begin{aligned}
 I &\leq \delta^{-\beta} \int_{D(z', 2\delta)} |\zeta - z'|^\alpha dm(\zeta) + \int_{\mathbf{C} \setminus D(z', 2\delta)} |\zeta - z'|^{\alpha-\beta} dm(\zeta) \\
 (2.2) \quad &\leq 2\pi\delta^{-\beta} \int_0^{2\delta} r^{\alpha+1} dr + 2\pi \int_{2\delta}^\infty r^{\alpha-\beta+1} dr \preceq \delta^{\alpha-\beta+2}.
 \end{aligned}$$

If  $|z' - z''| > \delta$ , then letting  $d := |z' - z''|$ ,

$$D' := D\left(z', \frac{d}{2}\right), \quad D'' := D\left(z'', \frac{d}{2}\right),$$

$$U_1 := D(z', 2d) \setminus (D' \cup D''), \quad U_2 := \mathbf{C} \setminus D(z', 2d),$$

and using the polar coordinates with centers at  $z'$  and  $z''$  respectively, we have

$$\begin{aligned}
 I &\preceq d^{-\beta} \int_{D'} |\zeta - z'|^\alpha dm(\zeta) + d^\alpha \int_{D''} (|\zeta - z''| + \delta)^{-\beta} dm(\zeta) \\
 &\quad + d^{\alpha-\beta} \int_{U_1} dm(\zeta) + \int_{U_2} |\zeta - z'|^{\alpha-\beta} dm(\zeta) \\
 &\leq 2\pi d^{-\beta} \int_0^{d/2} r^{\alpha+1} dr + 2\pi d^\alpha \int_0^{d/2} (r + \delta)^{-\beta+1} dr \\
 &\quad + d^{\alpha-\beta} \pi 4d^2 + 2\pi \int_{2d}^\infty r^{\alpha-\beta+1} dr \\
 (2.3) \quad &\preceq d^{-\beta+\alpha+2} + d^\alpha \delta^{2-\beta} \asymp d^\alpha \delta^{2-\beta}.
 \end{aligned}$$

Comparing (2.2) and (2.3) we obtain (2.1)

□

**Lemma 2** *Let  $0 < \delta < \varepsilon$ ,  $\alpha_j > -2$ ,  $j = 1, \dots, m$ , and  $\beta > 2 + \sum_{j=1}^m |\alpha_j|$ . Suppose that points  $z_1, \dots, z_m \in \mathbf{C}$  satisfy*

$$|z_j| < c, \quad |z_j - z_k| > 4\varepsilon, \quad j \neq k.$$

*Then, for any  $z \in \mathbf{C}$  with  $|z| < c$ , we have*

$$\begin{aligned}
 I^*(z) &:= \int_{D(0, c)} (|\zeta - z| + \delta)^{-\beta} \prod_{j=1}^m |\zeta - z_j|^{\alpha_j} dm(\zeta) \\
 (2.4) \quad &\leq c_2 \delta^{2-\beta} \prod_{j=1}^m (|z - z_j| + \delta)^{\alpha_j},
 \end{aligned}$$

where  $c_2 = c_2(\alpha_1, \dots, \alpha_m, z_1, \dots, z_m, \varepsilon, c, \beta)$ .

**Proof.** Let  $\alpha := \sum_{j=1}^m |\alpha_j|$  and

$$D_j := D(z_j, 2\varepsilon), D'_j := D(z_j, \varepsilon), \quad j = 1, \dots, m.$$

Consider two particular cases.

If  $z \notin \cup_{j=1}^m D_j$ , then using the polar coordinates with centers at  $z_j$  and  $z$  respectively we have

$$\begin{aligned} I^*(z) &\leq \sum_{j=1}^m \int_{D'_j} |\zeta - z_j|^{\alpha_j} dm(\zeta) + \int_{\mathbf{C}} (|\zeta - z| + \delta)^{-\beta} dm(\zeta) \\ (2.5) \quad &\leq \sum_{j=1}^m \int_0^\varepsilon r^{\alpha_j+1} dr + \int_0^\infty (r + \delta)^{-\beta+1} dr \leq \delta^{2-\beta}. \end{aligned}$$

If  $z \in D_k$  for some  $k = 1, \dots, m$ , then, applying Lemma 1, we obtain

$$\begin{aligned} I^*(z) &\leq \sum_{\substack{j=1 \\ j \neq k}}^m \int_{D'_j} |\zeta - z_j|^{\alpha_j} dm(\zeta) + \int_{\mathbf{C}} (|\zeta - z| + \delta)^{-\beta} |\zeta - z_k|^{\alpha_k} dm(\zeta) \\ (2.6) \quad &\leq \delta^{2-\beta} \prod_{j=1}^m (|z - z_j| + \delta)^{\alpha_j}. \end{aligned}$$

Comparing (2.5) and (2.6) we have (2.4). □

Now let  $z \in L = \partial G, 0 < r \leq \delta < (\text{diam } G)/4$ , and  $\alpha > -2$ . Since by the definition of a quasiconformal curve (1.3)

$$|G \cap C(z, r)| \succeq r,$$

where  $|S|$  means the *linear measure*, i.e. *length*, of  $S \subset \mathbf{C}$ , we have

$$(2.7) \quad \int_{G \cap D(z, \delta)} |\zeta - z|^\alpha dm(\zeta) = \int_0^\delta r^\alpha |G \cap C(z, r)| dr \succeq \delta^{\alpha+2}.$$

Therefore, if  $Z := \{z_1, \dots, z_m\} \subset L$  and  $\alpha_j > -2, j = 1, \dots, m$  are fixed, then for  $z \in L$  and  $\delta < \min_{j \neq k} |z_j - z_k|/4$ ,

$$(2.8) \quad \int_{G \cap D(z, \delta)} \prod_{j=1}^m |\zeta - z_j|^{\alpha_j} dm(\zeta) \geq \varepsilon_1 \delta^2 \prod_{j=1}^m (|z - z_j| + \delta)^{\alpha_j},$$

where  $\varepsilon_1 = \varepsilon_1(G, Z, \alpha_1, \dots, \alpha_m)$ .

Indeed, let  $d := d(z, Z) = |z - z_k|$  for some  $k = 1, \dots, m$ . If  $\delta \geq 2d$ , then by virtue of (2.7)

$$\begin{aligned} A &:= \int_{G \cap D(z, \delta)} \prod_{j=1}^m |\zeta - z_j|^{\alpha_j} dm(\zeta) \succeq \int_{G \cap D(z_k, \delta/2)} |\zeta - z_k|^{\alpha_k} dm(\zeta) \\ (2.9) \quad &\succeq \delta^{\alpha_k+2} \asymp \delta^2 \prod_{j=1}^m (\delta + |z - z_j|)^{\alpha_j} =: B. \end{aligned}$$

If  $\delta < 2d$ , then, according to (2.7),

$$(2.10) \quad A \succeq d^{\alpha_k} \int_{G \cap D(z, \delta/4)} dm(\zeta) \succeq \delta^2 d^{\alpha_k} \asymp B.$$

Comparing (2.9) and (2.10) we have (2.8).

Next, we introduce auxiliary families of quasiconformal curves and mappings as follows. Let  $K \geq 1$  be a coefficient of quasiconformality of  $L$ . It is well known (see [4, Chapter IV]) that the Riemann mapping function  $\Phi$  can be extended to a  $K^2$ -quasiconformal homeomorphism  $\Phi : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$ . Hence, each curve

$$L_\delta^* := \{z : |\Phi(z)| = 1 - \delta\}, \quad 0 \leq \delta < 1$$

is  $K^2$ -quasiconformal. Denote by  $\Omega_\delta^*$  the unbounded connected component of  $\overline{\mathbf{C}} \setminus L_\delta^*$ . The Riemann conformal mapping  $\Phi_\delta : \Omega_\delta^* \rightarrow \mathbf{D}^*$  with the normalization

$$\Phi_\delta(\infty) = \infty, \quad \Phi'_\delta(\infty) > 0$$

can be extended to a  $K^4$ -quasiconformal homeomorphism  $\Phi_\delta : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$ . Note that  $\Phi_0 = \Phi$ ,  $\Psi_0 = \Psi$ , and  $L_0^* = L$ . To study metric properties of  $\Phi_\delta$  and  $\Psi_\delta := \Phi_\delta^{-1}$ , we use the following statement.

**Lemma 3** (see [6, p. 97, Theorem 4.1] or [7, p. 29, Theorem 2.7]) *Suppose that  $F : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$  is a  $Q$ -quasiconformal mapping with  $Q \geq 1$  and  $F(\infty) = \infty$ . Assume also that  $\zeta_j \in \mathbf{C}$ ,  $w_j := F(\zeta_j)$ ,  $j = 1, 2, 3$ . Then:*

(i) *the conditions  $|\zeta_1 - \zeta_2| \leq c_3 |\zeta_1 - \zeta_3|$  and  $|w_1 - w_2| \leq c_4 |w_1 - w_3|$  are equivalent; besides, the constants  $c_3$  and  $c_4$  are mutually dependent and dependent on  $Q$ ;*

(ii) *if  $|\zeta_1 - \zeta_2| \leq c_3 |\zeta_1 - \zeta_3|$ , then*

$$c_5^{-1} \left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{1/Q} \leq \left| \frac{\zeta_1 - \zeta_3}{\zeta_1 - \zeta_2} \right| \leq c_5 \left| \frac{w_1 - w_3}{w_1 - w_2} \right|^Q,$$

where  $c_5 = c_5(c_3, Q)$ .

Let

$$\tilde{\zeta}_\delta := \Psi((1 + \delta)\Phi(\zeta)), \quad \zeta \in \overline{\Omega} \setminus \{\infty\}, \delta > 0.$$

For  $z \in L$  and  $\delta > 0$ , let a point  $z_\delta^* \in L_\delta$  satisfy  $|z - z_\delta^*| = \rho_\delta(z)$ . Applying Lemma 3 with  $F = \Phi$  and the triplet of points  $z, \tilde{z}_\delta, z_\delta^*$  we have

$$(2.11) \quad \rho_\delta(z) \asymp |z - \tilde{z}_\delta|, \quad z \in L.$$

Moreover, we claim that for  $z, \zeta \in L$  and  $0 < \delta \leq 1$ ,

$$(2.12) \quad |z - \tilde{\zeta}_\delta| \asymp |z - \zeta| + \rho_\delta(z).$$

Indeed, by Lemma 3 with  $F = \Phi$ ,

$$|z - \zeta| \preceq |z - \tilde{\zeta}_\delta| \quad \text{and} \quad |z - \tilde{z}_\delta| \preceq |z - \tilde{\zeta}_\delta|,$$

i.e.,

$$|z - \zeta| + \rho_\delta(z) \asymp |z - \zeta| + |z - \tilde{z}_\delta| \preceq |z - \tilde{\zeta}_\delta|.$$

Furthermore, the same Lemma 3 with  $F = \Phi$  also implies that if  $|\Phi(z) - \Phi(\zeta)| > \delta$  then

$$|z - \tilde{\zeta}_\delta| \asymp |z - \zeta| \succeq |z - \tilde{z}_\delta|$$

as well as if  $|\Phi(z) - \Phi(\zeta)| \leq \delta$  then

$$|z - \tilde{\zeta}_\delta| \asymp |z - \tilde{z}_\delta| \succeq |z - \zeta|.$$

That is, in both cases we have

$$|z - \tilde{\zeta}_\delta| \succeq |z - \zeta| + |z - \tilde{z}_\delta| \asymp |z - \zeta| + \rho_\delta(z)$$

which completes the proof of (2.12).

Next, for  $0 < v < u \leq 1$  and  $z \in L$ , Lemma 3 with  $F = \Phi$  and the triplet of points  $z, \tilde{z}_v, \tilde{z}_u$  as well as (2.11) imply

$$(2.13) \quad \left(\frac{u}{v}\right)^{1/K^2} \preceq \frac{\rho_u(z)}{\rho_v(z)} \preceq \left(\frac{u}{v}\right)^{K^2}.$$

For  $\xi \in \Omega \setminus \{\infty\}$ , let  $\xi_L := \Psi(\Phi(\xi)/|\Phi(\xi)|)$  and let  $\xi_* \in L$  satisfy  $d(\xi, L) = |\xi - \xi_*|$ . Applying Lemma 3 with  $F = \Phi$  and the triplet of points  $\xi, \xi_L, \xi_*$  we obtain

$$(2.14) \quad |\xi - \xi_L| \asymp d(\xi, L).$$

Therefore, for  $z \in \overline{G}$  and  $\xi \in \Omega \setminus \{\infty\}$ ,

$$\begin{aligned} |\xi - z| &\leq |\xi - \xi_L| + |\xi_L - z| \\ &\leq 2|\xi_L - \xi| + |\xi - z| \preceq |\xi - z|, \end{aligned}$$



i.e.,

$$(2.15) \quad |\xi - z| \asymp |\xi - \xi_L| + |\xi_L - z|.$$

Let for  $0 < \tau \leq 1$  and  $\zeta \in \overline{\Omega_\delta^*} \setminus \{\infty\}$ ,

$$\tilde{\zeta}_{\delta,\tau} := \Psi_\delta((1 + \tau)\Phi_\delta(\zeta)).$$

Lemma 3 with  $F = \Phi_\delta$  implies also that for  $z \in \overline{\Omega_\delta^*} \setminus \{\infty\}$  and  $\zeta \in \overline{\Omega_\delta^*}$  with  $|\zeta - z| \leq c_6|z - \tilde{\zeta}_{\delta,\tau}|$  we have

$$(2.16) \quad c_7^{-1}|z - \tilde{\zeta}_{\delta,\tau}| \leq |\zeta - \tilde{\zeta}_{\delta,\tau}| \leq c_7|z - \tilde{\zeta}_{\delta,\tau}|,$$

where  $c_7 = c_7(K, c_6) > 1$ .

Furthermore, let  $0 < \delta = \tau < 1/2$ . Since by [7, p. 376, Lemma 2.2] and Lemma 3,

$$(2.17) \quad |\Phi_\delta(z)| - 1 \asymp \delta, \quad z \in L,$$

(2.11) and (2.13), written for  $L_\delta^*$  and  $\Phi_\delta$  instead of  $L$  and  $\Phi$ , yield

$$(2.18) \quad d(z, L) \asymp |z - \tilde{z}_{\delta,\delta}|, \quad z \in L_\delta^*.$$

Moreover, we claim that

$$(2.19) \quad |\tilde{z}_{\delta,\delta} - z| \asymp |\tilde{z}_\delta - z|, \quad z \in L.$$

Indeed, let  $z^\bullet = z^\bullet(\delta) \in L_\delta^*$  satisfy  $|z - z^\bullet| = d(z, L_\delta^*)$ . Applying Lemma 3 with  $F = \Psi$  twice: first with the triplet  $\Phi(z), (1 - \delta)\Phi(z), \Phi(z^\bullet)$  and then with the triplet  $\Phi(z), (1 - \delta)\Phi(z), (1 + \delta)\Phi(z)$ , we obtain

$$(2.20) \quad |z - \tilde{z}_\delta| \asymp d(z, L_\delta^*).$$

Let  $z' = z'(\delta) := \Psi_\delta(\Phi_\delta(z)/|\Phi_\delta(z)|)$  so that by (2.14), written for  $\Omega_\delta^*$  instead of  $\Omega$ , we have  $d(z, L_\delta^*) \asymp |z - z'|$ . Since by (2.17) and Lemma 3 with  $F = \Psi_\delta$  and the triplet  $\Phi_\delta(z'), \Phi_\delta(z), (1 + \delta)\Phi_\delta(z)$

$$|z' - z| \asymp |z' - \tilde{z}'_{\delta,\delta}|,$$

according to (2.16)

$$(2.21) \quad |z - \tilde{z}_{\delta,\delta}| \asymp |z' - \tilde{z}'_{\delta,\delta}| \asymp |z' - z| \asymp d(z, L_\delta^*).$$

Comparing (2.20) and (2.21) we obtain (2.19).

To estimate the Christoffel function from above we use special polynomials defined as follows. For  $\xi \in \Omega \setminus \{\infty\}$ ,  $z \in \overline{G}$ , and  $n \in \mathbf{N}$  with  $n \geq 2$ , consider the *Dzjadyk kernel*  $K_{0,1,1,n}(\xi, z)$  associated with  $\overline{G}$  (see [11, p. 429] or [7, p. 387])

which is a polynomial in  $z$  of degree at most  $4n$  with coefficients depending on  $\xi$ . By virtue of [7, p. 389, Theorem 2.4] we have

$$(2.22) \quad \left| \frac{1}{\xi - z} - K_{0,1,1,n}(\xi, z) \right| \leq \frac{c_8}{|\xi - z|} \left| \frac{\tilde{\xi}_{1/n} - \xi}{\tilde{\xi}_{1/n} - z} \right|.$$

For  $\zeta \in L$ , define  $\xi = \xi(\zeta, n) := \tilde{\zeta}_{c_9/n}$ ,  $n > c_9$ , where  $c_9 > 1$  is chosen as follows. According to Lemma 3 with  $F = \Phi$  and the triplet  $\tilde{\xi}_{1/n}, \xi, \zeta$  as well as (2.14), for  $z \in \overline{G}$ ,

$$(2.23) \quad \left| \frac{\tilde{\xi}_{1/n} - \xi}{\tilde{\xi}_{1/n} - z} \right| = \left| \frac{\tilde{\xi}_{1/n} - \xi}{\tilde{\xi}_{1/n} - \zeta} \right| \left| \frac{\tilde{\xi}_{1/n} - \zeta}{\tilde{\xi}_{1/n} - z} \right| \leq \frac{c_{10}}{c_9^{1/K^2}} < \frac{1}{2c_8}$$

if  $c_9 := 1 + (2c_8c_{10})^{K^2}$ .

Since (2.22) and (2.23) imply for  $z \in \overline{G}$ ,

$$\left| \frac{1}{\xi - z} - K_{0,1,1,n}(\xi, z) \right| \leq \frac{1}{2|\xi - z|},$$

by (2.11), (2.13), and (2.15) we have

$$(2.24) \quad |K_{0,1,1,n}(\xi, z)| \asymp |\xi - z|^{-1} \asymp (|\zeta - z| + \rho_{1/n}(\zeta))^{-1}.$$

For  $\zeta \in L$  and any (fixed)  $s \in \mathbf{N}$ , consider polynomials (in  $z$ ) of degree at most  $4sn$  defined by

$$q_{n,s,\zeta}(z) := (\rho_{1/n}(\zeta) K_{0,1,1,n}(\xi(\zeta, n), z))^s, \quad Q_{n,s,\zeta}(z) := \frac{q_{n,s,\zeta}(\zeta, z)}{q_{n,s,\zeta}(\zeta, \zeta)}.$$

Summarizing, we let

$$p_{n,s,\zeta} := \begin{cases} 1 & \text{if } n \leq 8s, \\ Q_{\lfloor n/(4s) \rfloor, s, \zeta} & \text{if } n > 8s, \end{cases}$$

where  $\lfloor x \rfloor$  denotes the *integer part* of a real number  $x$ , and use (2.13) and (2.24) to obtain the following statement.

**Lemma 4** *For  $n \in \mathbf{N}$ ,  $\zeta \in L$  and fixed  $s \in \mathbf{N}$  there exists a polynomial  $p_{n,s,\zeta} \in \mathbf{P}_n$  with the following properties:*

(i)  $p_{n,s,\zeta}(\zeta) = 1$ ;

(ii) for  $z \in \overline{G}$ ,

$$|p_{n,s,\zeta}(z)| \leq c_{11} \left( \frac{\rho_{1/n}(\zeta)}{|\zeta - z| + \rho_{1/n}(\zeta)} \right)^s,$$

where  $c_{11} = c_{11}(G, s)$ .

### 3. Proof of Theorems

We start with a modification of the classical Ahlfors result [3]. As before, denote by  $K \geq 1$  a coefficient of quasiconformality of  $L$ .

**Lemma 5** (see [6, pp. 25-26, Lemma 1.4 and Corollary 1.3]). *There exists a quasiconformal reflection  $y : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$  with respect to  $L$  satisfying the following properties:*

$$(i) \ y(G) = \Omega, \ y(\Omega) = G,$$

$$y(z) = z, \quad z \in L;$$

(ii)  $y$  has continuous partial derivatives of first order in  $\mathbf{C} \setminus (L \cup \{z_0\})$ , where  $z_0 := y(\infty)$ ;

(iii) for  $\zeta_1, \zeta_2 \in \overline{G} \setminus D_0$ , where  $D_0 := D(z_0, d(z_0, L)/2)$ , the inequality

$$c_1^{-1} |\zeta_1 - \zeta_2| \leq |y(\zeta_1) - y(\zeta_2)| \leq c_1 |\zeta_1 - \zeta_2|$$

holds with  $c_1 = c_1(K) > 1$ ;

(iv) the inequalities

$$|y_{\overline{\zeta}}(\zeta)| \leq c_2 |y(\zeta)|^2, \quad \zeta \in D_0,$$

$$|y_{\overline{\zeta}}(\zeta)| \leq c_2, \quad \zeta \in G \setminus D_0,$$

hold with  $c_2 = c_2(K)$ .

Next, we claim that

$$(3.1) \quad |z - \zeta| + d(z, L) \preceq |z - y(\zeta)|, \quad z, \zeta \in G \setminus D_0.$$

Indeed, in the nontrivial case where  $d(z, L) < |z - \zeta|$ , we introduce a point  $z' \in L$  such that  $d(z, L) = |z - z'|$  and use Lemma 5 to obtain

$$\begin{aligned} |z - \zeta| + d(z, L) &< 2|z - \zeta| \leq 2|z - z'| + 2|z' - \zeta| \preceq |z - z'| + |z' - y(\zeta)| \\ &\leq 2|z - z'| + |z - y(\zeta)| \leq 3|z - y(\zeta)| \end{aligned}$$

which proves (3.1).

For the weight function  $h$  defined by (1.4) and  $1 \leq p < \infty$ , denote by  $A_p(h, G)$  the space of functions  $f$  analytic in  $G$  and satisfying

$$\|f\|_{A_p(h, G)}^p := \int_G |f|^p h dm < \infty.$$

Note that polynomials are in  $A_p(h, G)$ .

**Lemma 6** For  $p_n \in \mathbf{P}_n, n \in \mathbf{N}, 1 \leq p < \infty$ , and  $z \in G \setminus D_0$ , we have

$$(3.2) \quad |p'_n(z)| \leq c_3 d(z, L)^{-1-2/p} \prod_{j=1}^m |z - z_j|^{-\alpha_j/p} \|p_n\|_{A_p(h, G)},$$

where  $c_3 = c_3(G, h, p)$ .

**Proof.** Consider an analytic in  $G$  function

$$H_p(z) := \prod_{j=1}^m (z - z_j)^{\alpha_j/p}.$$

Since  $\int_G |p_n H_p| dm < \infty$ , we can use the Belyi integral formula (see [9] or [6, p. 110, Theorem 4.4]) to obtain for  $z \in G \setminus D_0$

$$\begin{aligned} p'_n(z) H_p(z) &= -p_n(z) H'_p(z) + (p_n(z) H_p(z))' \\ &= -p_n(z) H'_p(z) - \frac{2}{\pi} \int_G \frac{p_n(\zeta) H_p(\zeta)}{(y(\zeta) - z)^3} y_{\bar{\zeta}}(\zeta) dm(\zeta) \\ (3.3) \quad &=: -A(z) - B(z). \end{aligned}$$

According to the mean-value property for a subharmonic function  $|p_n|^p$  (see [17, p. 46, Theorem 2.6.8(b)]), letting  $d := d(z, L)$  we have

$$|p_n(z)|^p \leq \frac{4}{\pi d^2} \int_{D(z, d/2)} |p_n|^p dm \preceq d^{-2} \prod_{j=1}^m |z - z_j|^{-\alpha_j} \|p_n\|_{A_p(h, G)}^p$$

which implies

$$\begin{aligned} |A(z)| &= |p_n(z)| |H_p(z)| \left| \frac{H'_p(z)}{H_p(z)} \right| \\ (3.4) \quad &\leq |p_n(z)| |H_p(z)| \sum_{j=1}^m \frac{|\alpha_j|}{p |z - z_j|} \preceq d^{-1-2/p} \|p_n\|_{A_p(h, G)}. \end{aligned}$$

To estimate  $|B(z)|$  we consider two particular cases.

If  $p = 1$ , then by Lemma 5 and (3.1),

$$\begin{aligned} |B(z)| &\preceq \int_{D_0} |p_n| h dm + d^{-3} \int_{G \setminus D_0} |p_n| h dm \\ (3.5) \quad &\preceq d^{-3} \|p_n\|_{A_1(h, G)}. \end{aligned}$$

If  $p > 1$ , then Hölder's inequality with  $q := p/(p-1)$  yields

$$|B(z)| \preceq \|p_n\|_{A_p(h,G)} \left( \int_G \frac{|y_{\bar{\zeta}}(\zeta)|^q dm(\zeta)}{|y(\zeta) - z|^{3q}} \right)^{1/q} =: \|p_n\|_{A_p(h,G)} C(z)^{1/q}.$$

According to Lemma 2 with  $\beta = 3q, \delta = d$  and  $\alpha_j = 0$  as well as Lemma 5 and (3.1) we obtain

$$C(z) \preceq \int_{D_0} dm + \int_{G \setminus D_0} \frac{dm(\zeta)}{(|\zeta - z| + d)^{3q}} \preceq d^{2-3q}$$

which implies

$$(3.6) \quad |B(z)| \preceq d^{-1-2/p} \|p_n\|_{A_p(h,G)}.$$

Comparing (3.3)-(3.6) we have (3.2). □

**Lemma 7** *There exists  $\varepsilon = \varepsilon(G)$  such that for  $p_n \in \mathbf{P}_n, n \in \mathbf{N}, 1 \leq p < \infty, z \in L$ , and  $\zeta \in D(z, \varepsilon \rho_{1/n}(z))$  the inequality*

$$(3.7) \quad |p'_n(\zeta)| \leq c_4 \rho_{1/n}(z)^{-1-2/p} \prod_{j=1}^m (\rho_{1/n}(z) + |z - z_j|)^{-\alpha_j/p} \|p_n\|_{A_p(h,G)}$$

holds with  $c_4 = c_4(\varepsilon, G, h, p)$ .

**Proof.** Without loss of generality we assume that  $n > 2$  and, in addition to the points  $z_j \in L, j = 1, \dots, m$ , we introduce points

$$z_{j,n} := \Psi \left( \left( 1 - \frac{2}{n} \right) \Phi(z_j) \right)$$

which, according to Lemma 3 with  $F = \Phi$  and the triplet of points  $z, z_j, z_{j,n}$  satisfy

$$(3.8) \quad |z - z_j| \asymp |z - z_{j,n}|, \quad z \in L_{1/n}^*.$$

Let  $1 \leq k \leq n$  be the degree of  $p_n$  and let

$$\zeta_n := \tilde{\zeta}_{1/n, 1/n} = \Psi_{1/n} \left( \left( 1 + \frac{1}{n} \right) \Phi_{1/n}(\zeta) \right), \quad \zeta \in \Omega_{1/n}^*.$$

Consider subharmonic in  $\Omega_{1/n}^*$  function

$$\begin{aligned} f(\zeta) = f_{n,p}(\zeta) &:= \ln |p'_n(\zeta)| + \left( 1 + \frac{2}{p} \right) \ln |\zeta - \zeta_n| + \frac{1}{p} \sum_{j=1}^m \alpha_j \ln |\zeta - z_{j,n}| \\ &\quad - \left( k + \frac{2}{p} + \frac{1}{p} \sum_{j=1}^m \alpha_j \right) \ln |\Phi_{1/n}(\zeta)| \end{aligned}$$

(which is harmonic and bounded in a neighborhood of infinity).

Since by virtue of (2.18), Lemma 6, and (3.8),

$$f(\zeta) \leq c_5 + \ln \|p_n\|_{A_p(h,G)}, \quad \zeta \in L_{1/n}^*,$$

by the maximum principle (see [17, p. 29]) the same inequality holds for  $\zeta \in D(z, \varepsilon \rho_{1/n}(z))$ , where  $z \in L$  and  $\varepsilon$  is chosen such that

$$\varepsilon \rho_{1/n}(z) \leq \frac{1}{2} d(z, L_{1/n}^*).$$

The existence of such a constant  $\varepsilon$  is guaranteed by Lemma 3 with  $F = \Phi$  and (2.11).

Applying (2.12) and Lemma 3 with  $F = \Phi$  and the triplet  $z, z_{j,n}, (\tilde{z}_j)_{1/n}$  we obtain for  $\zeta \in D(z, \varepsilon \rho_{1/n}(z))$

$$|\zeta - z_{j,n}| \asymp |z - z_{j,n}| \asymp |z - (\tilde{z}_j)_{1/n}| \asymp \rho_{1/n}(z) + |z - z_j|.$$

Therefore, according to (2.11), (2.16), (2.17), and (2.19) we further have

$$\begin{aligned} \ln |p'_n(\zeta)| &= f(\zeta) - \left(1 + \frac{2}{p}\right) \ln |\zeta - \zeta_n| - \frac{1}{p} \sum_{j=1}^m \alpha_j \ln |\zeta - z_{j,n}| \\ &\quad + \left(k + \frac{2}{p} + \frac{1}{p} \sum_{j=1}^m \alpha_j\right) \ln |\Phi_{1/n}(\zeta)| \\ &\leq c_6 + \ln \|p_n\|_{A_p(h,G)} - \left(1 + \frac{2}{p}\right) \ln \rho_{1/n}(z) \\ &\quad - \frac{1}{p} \sum_{j=1}^m \alpha_j \ln (\rho_{1/n}(z) + |z - z_j|), \end{aligned}$$

which yields (3.7). □

**Proof of Theorem 1.** Let  $s \in \mathbf{N}$  be a fixed number with  $s > 2 + \sum_{j=1}^m |\alpha_j|$ . Consider polynomial  $p_n = p_{n,s,z}$  from Lemma 4. On account of (1.1), Lemma 2 with  $\beta := sp$ ,  $\delta := \rho_{1/n}(z)$  and Lemma 4, we have

$$\begin{aligned} \lambda_n(\nu, p, z) &\leq \int_G |p_n|^p h dm \\ &\leq \rho_{1/n}(z)^{sp} \int_G (|z - \zeta| + \rho_{1/n}(z))^{-sp} \prod_{j=1}^m |\zeta - z_j|^{\alpha_j} dm(\zeta) \end{aligned}$$

$$\preceq \rho_{1/n}(z)^2 \prod_{j=1}^m (|z - z_j| + \rho_{1/n}(z))^{\alpha_j},$$

which proves the right-hand side of (1.5).

In order to prove the left-hand side of (1.5), it is sufficient to show that for any  $z \in L$  and  $p_n \in \mathbf{P}_n$  with  $p_n(z) = 1$ , we have

$$(3.9) \quad \int_G |p_n|^p h dm \succeq \rho_{1/n}(z)^2 \prod_{j=1}^m (|z - z_j| + \rho_{1/n}(z))^{\alpha_j}.$$

In the nontrivial case where

$$\int_G |p_n|^p h dm \leq \rho_{1/n}(z)^2 \prod_{j=1}^m (|z - z_j| + \rho_{1/n}(z))^{\alpha_j}$$

Lemma 7 implies

$$|p'_n(\zeta)| \leq \frac{c_4}{\rho_{1/n}(z)}, \quad \zeta \in D(z, \varepsilon \rho_{1/n}(z)).$$

Moreover, for the same  $\zeta$

$$|p_n(\zeta) - 1| \leq \int_{[z, \zeta]} |p'_n(\xi)| |d\xi| \leq c_4 \frac{|z - \zeta|}{\rho_{1/n}(z)}$$

and if

$$|z - \zeta| \leq \varepsilon_1 \rho_{1/n}(z), \quad \varepsilon_1 := \min \left( \varepsilon, \frac{1}{2c_4} \right),$$

we obtain  $|p_n(\zeta)| \geq 1/2$ .

Hence, according to (2.8) with  $\delta = \varepsilon_2 \rho_{1/n}(z)$ , where  $\varepsilon_2 = \varepsilon_2(G) < \varepsilon_1$  is chosen so that  $\delta < \min_{j \neq k} |z_j - z_k|/4$ , we have

$$\begin{aligned} \int_G |p_n|^p h dm &\geq \int_{G \cap D(z, \varepsilon_2 \rho_{1/n}(z))} |p_n|^p h dm \\ &\succeq \rho_{1/n}(z)^2 \prod_{j=1}^m (|z - z_j| + \rho_{1/n}(z))^{\alpha_j} \end{aligned}$$

which proves (3.9).

□

**Proof of Theorem 2.** By Lemma 4 with the quasidisk

$$G = \{z = x + iy : 0 < x < 1, |y| < e^{-1}\}$$

for any fixed  $k \in \mathbf{N}$  and any  $n \in \mathbf{N}$  there exists a polynomial  $p_n = p_{n,4k,0} \in \mathbf{P}_n$  such that  $p_n(0) = 1$  and

$$|p_n(\zeta)| \preceq \begin{cases} 1 & \text{if } \zeta \in G, |\zeta| \leq 1/n, \\ (n|\zeta|)^{-4k} & \text{if } \zeta \in G, |\zeta| > 1/n. \end{cases}$$

Since  $G^* \subset G$  and by the Löwner inequality [14] (see also [7, p. 359, Corollary 2.5]),

$$\rho_{1/n}(0) \geq \frac{\text{diam } G^*}{8n^2},$$

by virtue of (1.1) we obtain

$$\begin{aligned} \lambda_n(m^*, p, 0) &\leq \int_{G^*} |p_n|^p dm^* \leq \int_{G^* \cap D(0, 1/n)} |p_n|^p dm^* + \int_{G^* \setminus D(0, 1/n)} |p_n|^p dm^* \\ &\preceq \frac{1}{n} e^{-n} + n^{-4kp} \int_{1/(2n)}^1 x^{-4kp} e^{-1/x} dx \preceq n^{-4k} \preceq \rho_{1/n}(0)^{2k} \end{aligned}$$

from which (1.10) follows. □

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V. V. Andrievskii

Department of Mathematical Sciences

Kent State University

Kent, OH 44242

USA

e-mail: andriyev@math.kent.edu

tel: 330-672-9029